

NUMERICAL CHARACTERISATION OF QUADRICS

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ABSTRACT. Let X be a Fano manifold such that $-K_X \cdot C \geq \dim X$ for every rational curve $C \subset X$. We prove that X is a projective space or a quadric.

INTRODUCTION

Let X be a Fano manifold, i.e. a projective manifold with ample anticanonical bundle. If the Picard number of X is at least two, Mori theory shows the existence of at least two non-trivial morphisms $\varphi_i : X \rightarrow Y_i$ which contain some interesting information on the geometry of X . On the contrary, when the Picard number equals one Mori theory does not yield any information, and one is thus led to studying X in terms of the positivity of the anticanonical bundle. A well-known example of such a characterisation is the following theorem of Kobayashi–Ochiai.

A. Theorem [KO73]. *Let X be a projective manifold of dimension n . Suppose that $-K_X \simeq dH$ with H an ample line bundle on X .*

- a) *If $d = n + 1$, then $X \simeq \mathbb{P}^n$.*
- b) *If $d = n$, then $X \simeq \mathbb{Q}^n$.*

(Throughout the paper, \mathbb{Q}^n designates a smooth quadric hypersurface in \mathbb{P}^{n+1} for any positive integer n .)

The divisibility of $-K_X$ in the Picard group is a rather restrictive condition, so it is natural to ask for similar characterisations under (a priori) weaker assumptions. Based on Kebekus' study of singular rational curves [Keb02b], Cho, Miyaoka and Shepherd-Barron proved a generalisation of the first part of Theorem A:

B. Theorem [CMSB02, Keb02a]. *Let X be a Fano manifold of dimension n . Suppose that*

$$-K_X \cdot C \geq n + 1 \quad \text{for all rational curves } C \subset X.$$

Then $X \simeq \mathbb{P}^n$.

The aim of this paper is to prove a similar generalisation for the second part of Theorem A:

C. Theorem. *Let X be a Fano manifold of dimension n . Suppose that*

$$-K_X \cdot C \geq n \quad \text{for all rational curves } C \subset X.$$

Then $X \simeq \mathbb{P}^n$ or $X \simeq \mathbb{Q}^n$.

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This statement already appears in a paper of Miyaoka [Miy04, Thm.0.1], but the proof there has a gap (cf. Remark 4.2). In this paper we borrow some ideas and tools from Miyaoka's, yet give a proof based on a completely different strategy. Note also that Hwang gave a proof under the additional assumption that the general VMRT (see below) is smooth [Hwa13, Thm.1.11], a property that does not hold for every Fano manifold [CD15, Thm.1.10].

In the proof of Theorem C, we have to assume $n \geq 4$; for $n \leq 3$ the statement follows easily from classification results.

The assumption that X is Fano assures that $\rho(X) = 1$ because of the Ionescu–Wiśniewski inequality [Ion86, Thm.0.4], [Wiś91, Thm.1.1] (see §3.1). It is possible to remove this assumption: the Ionescu–Wiśniewski inequality together with [HN13, Thm.1.3] enable one to deal with the case $\rho(X) > 1$, and one gets the following.

D. Corollary. *Let X be a projective manifold of dimension n containing a rational curve. If*

$$-K_X \cdot C \geq n \quad \text{for all rational curves } C \subset X,$$

then X is a projective space, a hyperquadric, or a projective bundle over a curve.

(Note that under the assumptions of Corollary D, if $\rho(X) = 1$ then X is Fano.)

Outline of the proof. In the situation of Theorem C let \mathcal{K} be a family of minimal rational curves on X . By Mori's bend-and-break lemma a minimal curve $[l] \in \mathcal{K}$ satisfies $-K_X \cdot l \leq n + 1$ and if equality holds then $X \simeq \mathbb{P}^n$ by [CMSB02]. By our assumption we are thus left to deal with the case $-K_X \cdot l = n$. For a general point $x \in X$ the space \mathcal{K}_x parametrising curves in \mathcal{K} passing through x then has dimension $n - 2$ and by [Keb02b, Thm.3.4] there exists a morphism

$$\tau_x : \mathcal{K}_x \rightarrow \mathbb{P}(\Omega_{X,x})$$

which maps a general curve $[l] \in \mathcal{K}_x$ to its tangent direction $T_{l,x}^\perp$. By [HM04, Thm.1] this map is birational onto its image \mathcal{V}_x , the *variety of minimal rational tangents* (VMRT) at x . We denote by $\mathcal{V} \subset \mathbb{P}(\Omega_X)$ the total VMRT, i.e. the closure of the locus covered by the VMRTs \mathcal{V}_x for $x \in X$ general. For the proof of Theorem C we will compute the cohomology class of the total VMRT $\mathcal{V} \subset \mathbb{P}(\Omega_X)$ in terms of the tautological class ζ and π^*K_X , where $\pi : \mathbb{P}(\Omega_X) \rightarrow X$ is the projection map. This computation is based on the construction, on the manifold X , of a family \mathcal{W}° of smooth rational curves such that for every $[C] \in \mathcal{W}^\circ$ one has

$$T_X|_C \simeq \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus n};$$

it lifts to a family on $\mathbb{P}(\Omega_X)$ by associating to a curve $C \subset X$ the image \tilde{C} of the morphism $C \rightarrow \mathbb{P}(\Omega_X)$ defined by the invertible quotient

$$\Omega_X|_C \rightarrow \Omega_C.$$

The main technical statement of this paper is:

E. Proposition. *Let $X \not\simeq \mathbb{P}^n$ be a Fano manifold of dimension $n \geq 4$, and suppose that*

$$-K_X \cdot C \geq n \quad \text{for all rational curves } C \subset X.$$

Then, in the above notation, one has $\mathcal{V} \cdot \tilde{C} = 0$ for all $[C] \in \mathcal{W}_0$.

Once we have shown this statement a similar intersection computation involving a general minimal rational curve l yields that the VMRT $\mathcal{V}_x \subset \mathbb{P}(\Omega_{X,x})$ is a hypersurface of degree at most two. We then conclude with some earlier results of Araujo, Hwang, and Mok [Ara06, Hwa07, Mok08].

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1. NOTATION AND CONVENTIONS

We work over the field \mathbb{C} of complex numbers. Topological notions refer to the Zariski topology.

We use the modern notation for projective spaces, as introduced by Grothendieck: if \mathcal{E} is a locally free sheaf on a scheme X , we let $\mathbb{P}(\mathcal{E})$ be $\mathbf{Proj}(\mathrm{Sym} \mathcal{E})$. If L is a line in a vector space V , L^\perp designates the corresponding point in $\mathbb{P}(V^\vee)$.

A variety is an integral scheme of finite type over \mathbb{C} , a manifold is a smooth variety. A fibration is a proper surjective morphism with connected fibres $\varphi : X \rightarrow Y$ such that X and Y are normal and $\dim X > \dim Y > 0$.

We will use the standard terminology and results on rational curves, as explained in [Kol96, Ch.II], [Deb01, Ch.2,3,4], and [Hwa01]. Let X be a variety. We remind the reader that following [Kol96, II, Def.2.11], the notation $\mathrm{RatCurves}^n X$ refers to the union of the normalisations of those locally closed subsets of the Chow variety of X parametrising irreducible rational curves (the superscript n is a reminder that we normalised, and has nothing to do with the dimension).

For technical reasons, we have to consider families of rational curves on X as living alternately in $\mathrm{RatCurves}^n X$ and in $\mathrm{Hom}(\mathbb{P}^1, X)$. Our general policy is to call $\mathrm{Hom}_{\mathcal{R}} \subset \mathrm{Hom}(\mathbb{P}^1, X)$ the family corresponding to $\mathcal{R} \subset \mathrm{RatCurves}^n X$.

2. PRELIMINARIES ON CONIC BUNDLES

In this section, we establish some basic facts about conic bundles over a curve and compute some intersection numbers which will turn out to be crucial for the proof of Proposition E. All these statements appear in one form or another in [Miy04, §2], but we recall them and their proofs for the clarity of exposition.

2.1. Definition. *A conic bundle is an equidimensional projective fibration $\varphi : X \rightarrow Y$ such that there exists a rank three vector bundle $V \rightarrow Y$ and an embedding $X \hookrightarrow \mathbb{P}(V)$ that maps every φ -fibre $\varphi^{-1}(y)$ onto a conic (i.e. the zero scheme of a degree 2 form) in $\mathbb{P}(V_y)$. The set*

$$\Delta := \{y \in Y \mid \varphi^{-1}(y) \text{ is not smooth}\}$$

is called the discriminant locus of the conic bundle.

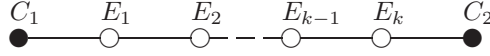
2.2. Lemma. *Let S be a smooth surface admitting a projective fibration $\varphi : S \rightarrow T$ onto a smooth curve such that the general fibre is \mathbb{P}^1 , and such that $-K_S$ is φ -nef. Let F be a reducible φ -fibre and suppose that*

$$F = C_1 + C_2 + F',$$

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where the C_i are (-1) -curves and $C_i \not\subset F'$. Then $F' = \sum E_j$ is a reduced chain of (-2) -curves and the dual graph of F is as depicted in Figure 1.

FIGURE 1.



Proof. Write $F' = \sum_{j=1}^k a_j E_j$, $a_j \in \mathbb{N}$, where E_1, \dots, E_k are the irreducible components of F' . Note first that since $-K_S \cdot F = 2$ and $-K_S \cdot C_i = 1$, the fact that $-K_S$ is φ -nef implies $-K_S \cdot E_j = 0$ for all j . Since E_j is an irreducible component of a reducible fibre, we have $E_j^2 < 0$. Thus we see that each E_j is a (-2) -curve.

We will now proceed by induction on the number of irreducible components of F' , the case $F' = 0$ being trivial. Let $\mu : S \rightarrow S'$ be the blow-down of the (-1) -curve C_2 ; then by the rigidity lemma [Deb01, Lemma 1.15], there is a morphism $\varphi' : S' \rightarrow T$ such that $\varphi = \varphi' \circ \mu$. Note that S' is smooth and $-K_{S'}$ is φ' -nef (see, e.g., [CH14, Rem.5.7]). We also have

$$0 = C_2 \cdot F = -1 + C_2 \cdot (C_1 + \sum_{i=1}^k a_i E_i),$$

so C_2 meets $C_1 + \sum_{i=1}^k a_i E_i$ transversally in exactly one point. If $C_2 \cdot C_1 > 0$, then $\mu_*(C_1)$ has self-intersection 0, yet it is also an irreducible component of the reducible fibre $\mu_*(C_1 + \sum_{i=1}^k a_i E_i)$, a contradiction. Thus (up to renumbering) we can suppose that $C_2 \cdot E_1 = 1$ and $a_1 = 1$. In particular $\mu_*(E_1)$ is a (-1) -curve, so

$$\mu_*(C_1 + \sum_{i=1}^k a_i E_i) = \mu_*(C_1) + \mu_*(E_1) + \mu_*(\sum_{i=2}^k a_i E_i)$$

satisfies the induction hypothesis. \square

In the following we will use that for every normal surface one can define an intersection theory using the Mumford pull-back to the minimal resolution, cf. [Sak84].

2.3. Lemma. *Let S be a normal surface admitting a projective fibration $\varphi : S \rightarrow T$ onto a smooth curve such that the general fibre is \mathbb{P}^1 and such that every fibre is reduced and has at most two irreducible components. Then*

- a) φ is a conic bundle;
- b) S has at most A_k -singularities; and
- c) if $s \in S_{\text{sing}}$, then $s = F_{\varphi(s),1} \cap F_{\varphi(s),2}$ where $F_{\varphi(s)} = F_{\varphi(s),1} + F_{\varphi(s),2}$ is the decomposition of the fibre over $\varphi(s)$ in its irreducible components. In particular $F_{\varphi(s)}$ is a reducible conic.

Proof. If a fibre $\varphi^{-1}(t)$ is irreducible, then φ is a \mathbb{P}^1 -bundle in a neighbourhood of $\varphi^{-1}(t)$ [Kol96, II, Thm.2.8]. Thus we only have to consider points $t \in T$ such that $S_t := \varphi^{-1}(t)$ is reducible. Since $p_a(S_t) = 0$ and $S_t = C_1 + C_2$ is reduced, we see

that S_t is a union of two \mathbb{P}^1 's meeting transversally in a point. Since $S_t = \varphi^*t$ is a Cartier divisor, this already implies c).

Let $\varepsilon : \hat{S} \rightarrow S$ be the minimal resolution of the singular points lying on S_t . Then we have

$$K_{\hat{S}} \equiv \varepsilon^* K_S - E,$$

with E an effective ε -exceptional \mathbb{Q} -divisor. Denote by \hat{C}_i the proper transform of C_i . If $K_{\hat{S}} \cdot \hat{C}_i < -1$, then \hat{C}_i deforms in \hat{S} [Kol96, II, Thm.1.15]. Yet \hat{C}_i is an irreducible component of a reducible $\varphi \circ \varepsilon$ -fibre, so this is impossible. So we have

$$K_S \cdot C_i \geq K_{\hat{S}} \cdot \hat{C}_i \geq -1$$

for $i = 1, 2$. Since $K_S \cdot (C_1 + C_2) = -2$, this implies that $K_S \cdot C_i = -1$ and $E = 0$. Thus S has canonical singularities. Since canonical surface singularities are Gorenstein we see that $-K_S$ is Cartier, φ -ample and defines an embedding

$$S \subset \mathbb{P}(V := \varphi_*(\mathcal{O}_S(-K_S)))$$

into a \mathbb{P}^2 -bundle mapping each fibre onto a conic. This proves a).

Since ε is crepant, the divisor $-K_{\hat{S}}$ is $\varphi \circ \varepsilon$ -nef. Moreover the proper transforms \hat{C}_i are (-1) -curves in \hat{S} . By Lemma 2.2 this proves b). \square

The following fundamental lemma should be seen as an analogue of the basic fact that a projective bundle over a curve contains at most one curve with negative self-intersection.

2.4. Lemma [Miy04, Prop.2.4]. *Let S be a normal projective surface that is a conic bundle $\varphi : S \rightarrow T$ over a smooth curve T , and denote by Δ the discriminant locus. Suppose that φ has two disjoint sections σ_1 and σ_2 , both contained in the smooth locus of S . Suppose moreover that for every $t \in \Delta$, the fibre F_t has a decomposition $F_t = F_{t,1} + F_{t,2}$ such that*

$$(C1) \quad \sigma_i \cdot F_{t,j} = \delta_{i,j}.$$

Eventually, assume that there exists a nef and big divisor H on S such that

$$(C2) \quad H \cdot \sigma_1 = H \cdot \sigma_2 = 0.$$

Let $\varepsilon : \hat{S} \rightarrow S$ be the minimal resolution. Let σ be a φ -section, and $\hat{\sigma} \subset \hat{S}$ its proper transform. Then the following holds:

- a) *If $(\hat{\sigma})^2 < 0$, then $\sigma = \sigma_1$ or $\sigma = \sigma_2$.*
- b) *If $(\hat{\sigma})^2 = 0$ then σ is disjoint from $\sigma_1 \cup \sigma_2$.*

Remark. In the situation above the conic bundle does not have any non-reduced fibre, since there exists a section that is contained in the smooth locus.

Proof. Preparation: contraction to a smooth ruled surface. Lemma 2.3 applies to the surface S . It follows that S has an A_{k_t} -singularity ($k_t \geq 0$) in $F_{t,1} \cap F_{t,2}$ for every $t \in \Delta$, and no further singularity. In particular, the dual graph of $(\varphi \circ \varepsilon)^{-1}(t)$ is as described in Lemma 2.2 for every $t \in \Delta$.

We consider the birational morphism

$$\hat{\mu} : \hat{S} \rightarrow S^b$$

defined as the composition, for every $t \in \Delta$, of the blow-down of the proper transform $\hat{F}_{t,1}$ of $F_{t,1}$ and of all the k_t (-2) -curves contained in $(\varphi \circ \varepsilon)^{-1}(t)$. Since $\hat{\mu}$ is a composition of blow-down of (-1) -curves, the surface S^b is smooth. By the rigidity lemma [Deb01, Lemma 1.15], there is a morphism $\varphi^b : S^b \rightarrow T$. All its fibres are irreducible rational curves, so it is a \mathbb{P}^1 -bundle by [Kol96, II, Thm.2.8]. Again by the rigidity lemma, $\hat{\mu}$ factors through ε , i.e. there is a birational morphism $\mu : S \rightarrow S^b$ such that $\hat{\mu} = \mu \circ \varepsilon$; it is the contraction of all the curves $F_{t,1}$, $t \in \Delta$.

Since σ_1 meets $F_{t,1}$ in a smooth point of S , the proper transforms $\hat{\sigma}_1$ and $\hat{F}_{t,1}$ meet in the same point. Thus (the successive images of) $\hat{\sigma}_1$ meets the exceptional divisor of all the blow-downs of (-1) -curves composing $\hat{\mu}$, and since the section $\sigma_1^b := \hat{\mu}(\hat{\sigma}_1)$ is smooth, all the intersections are transversal. Vice versa we can say that \hat{S} is obtained from S^b by blowing up points on (the proper transforms of) σ_1^b .

By the symmetry condition (C1) the curve σ_2 is disjoint from the μ -exceptional locus, so if we set $\sigma_2^b := \mu(\sigma_2)$, then we have $(\sigma_2^b)^2 = (\sigma_2)^2$. Since $H \cdot \sigma_2 = 0$ the Hodge index theorem implies $(\sigma_2^b)^2 = \sigma_2^2 < 0$. In the notation of [Har77, V, Ch.2] $\varphi^b : S^b \rightarrow T$ is a ruled surface with invariant $-e := (\sigma_2^b)^2 > 0$. In particular the Mori cone $\overline{\text{NE}}(S^b)$ is generated by a general φ^b -fibre F and σ_2^b . Since $\sigma_1^b \cdot \sigma_2^b = 0$ and $\sigma_1^b \cdot F = 1$, we have

$$(2.4.1) \quad \sigma_1^b \equiv \sigma_2^b + eF.$$

Conclusion. Let now $\sigma \subset S$ be a section that is distinct from both σ_1 and σ_2 . Then $\sigma^b := \mu(\sigma)$ is distinct from both σ_1^b and σ_2^b . Since $\sigma^b \neq \sigma_2^b$ we have

$$(2.4.2) \quad \sigma^b \equiv \sigma_2^b + cF$$

for some $c \geq e$ [Har77, V, Prop.2.20]. Since $\sigma^b \neq \sigma_1^b$ we have

$$(2.4.3) \quad \sigma^b \cdot \sigma_1^b \geq \sum_{t \in \Delta} \tau_t,$$

where τ_t is the intersection multiplicity of σ^b and σ_1^b at the point $F_t \cap \sigma_1^b$. Denote by $\hat{\sigma} \subset \hat{S}$ the proper transform of $\sigma \subset S$, which is also the proper transform of $\sigma^b \subset S^b$. By our description of $\hat{\mu}$ as a sequence of blow-ups in σ_1^b we obtain

$$(\hat{\sigma})^2 = (\sigma^b)^2 - \sum_{t \in \Delta} \min(\tau_t, k_t + 1) \geq (\sigma^b)^2 - \sum_{t \in \Delta} \tau_t.$$

By (2.4.3) this implies

$$(\hat{\sigma})^2 \geq (\sigma^b)^2 - \sigma^b \cdot \sigma_1^b = \sigma^b \cdot (\sigma^b - \sigma_1^b).$$

Plugging in (2.4.1) and (2.4.2) we obtain

$$(2.4.4) \quad (\hat{\sigma})^2 \geq c - e \geq 0.$$

This shows statement a).

Suppose now that $(\hat{\sigma})^2 = 0$. Then by (2.4.4) we have $c = e$, hence $\sigma^b \cdot \sigma_2^b = 0$. Being distinct, the two curves σ^b and σ_2^b are therefore disjoint, and so are their proper transforms $\hat{\sigma}$ and $\hat{\sigma}_2$. Note now that ε is an isomorphism in a neighbourhood of $\hat{\sigma}_2$, so $\sigma = \varepsilon(\hat{\sigma})$ is disjoint from $\sigma_2 = \varepsilon(\hat{\sigma}_2)$. In order to see that σ and σ_1 are disjoint, we repeat the same argument but contract those fibre components which meet σ_2 . This proves statement b). \square

3. THE MAIN CONSTRUCTION

3.1. Set-up. For the whole section, we let $X \not\cong \mathbb{P}^n$ be a Fano manifold of dimension $n \geq 4$, and suppose that

$$(3.1.1) \quad -K_X \cdot C \geq n \quad \text{for all rational curves } C \subset X.$$

This is the situation of Proposition E; let us show that it implies that the Picard number $\rho(X)$ equals 1.

The Mori cone $\overline{\text{NE}}(X)$ generates $\text{NS}(X)$ as a vector space, and since X is Fano the number of extremal rays of $\overline{\text{NE}}(X)$ is therefore at least $\rho(X)$. Now the Ionescu–Wiśniewski inequality [Ion86, Thm.0.4], [Wiś91, Thm.1.1] implies that the contraction of any extremal ray of $\overline{\text{NE}}(X)$ is of fibre type, with a base of dimension at most 1. If $\rho(X) > 1$, there are therefore at least two Mori fibrations, with bases of dimension 1; but then their respective fibres are divisors in X , and since $n > 2$ their intersection contains a curve that is contracted by both fibrations, a contradiction.

Recall that a family of *minimal rational curves* is an irreducible component \mathcal{K} of $\text{RatCurves}^n(X)$ such that the curves in \mathcal{K} dominate X , and for $x \in X$ general the algebraic set $\mathcal{K}_x^b \subset \mathcal{K}$ parametrising curves passing through x is proper. We will use the following simple observation:

3.2. Lemma. *In the situation of Proposition E, let $l \subset X$ be a rational curve such that $-K_X \cdot l = n$. Then any irreducible component \mathcal{K} of $\text{RatCurves}^n X$ containing $[l]$ is a family of minimal rational curves.*

Proof. Condition (3.1.1) implies the properness of \mathcal{K} [Kol96, II, (2.14)], so we only have to show that the deformations of l dominate X . We have $\dim \mathcal{K} \geq 2n - 3$ by [Kol96, II, Thm.1.2, Thm.2.15]. Thus if \mathcal{K} is not covering, then for a point $x \in X$ such that $\mathcal{K}_x^b \neq \emptyset$ one has $\dim \mathcal{K}_x^b \geq n - 1$. Yet by the bend-and-break lemma [Deb01, Prop.3.2] the universal family over \mathcal{K}_x^b has a generically finite map to X . Thus the curves in \mathcal{K}_x^b dominate X , a contradiction. \square

3.3. Minimal rational curves and VMRTs. Since X is Fano, it contains a rational curve l . Since $X \not\cong \mathbb{P}^n$, there exists a rational curve with $-K_X \cdot l = n$ by [CMSB02], and by Lemma 3.2 there exists a family of minimal rational curves containing the point $[l] \in \text{RatCurves}^n(X)$. We fix once and for all such a family, which we call \mathcal{K} .

For $x \in X$ general, denote by \mathcal{K}_x the normalisation of the algebraic set $\mathcal{K}_x^b \subset \mathcal{K}$ parametrising curves passing through x . Every member of \mathcal{K}_x^b is a free curve (this follows from the argument of [Kol96, II, proof of Thm.3.11]), so \mathcal{K}_x is smooth and has dimension $n - 2 \geq 2$ [Kol96, II, (1.7) and (2.16)].

By results of Kebekus, a general curve $[l] \in \mathcal{K}_x^b$ is smooth [Keb02b, Thm.3.3], and the *tangent map*

$$\tau_x : \mathcal{K}_x \rightarrow \mathbb{P}(\Omega_{X,x})$$

which to a general curve $[l]$ associates its tangent direction $T_{l,x}^\perp$ is a finite morphism [Keb02b, Thm.3.4]. Its image \mathcal{V}_x is called the *variety of minimal rational tangents* (VMRT) at x . The map τ_x is birational by [HM04, Thm.1], so the normalisation of \mathcal{V}_x is \mathcal{K}_x , which is smooth (this is [HM04, Cor.1]). Also, one can associate to a general point $v \in \mathcal{V}_x$ a unique minimal curve $[l] \in \mathcal{K}_x$. We denote by $\mathcal{V} \subset \mathbb{P}(\Omega_X)$

the *total VMRT*, i.e. the closure of the locus covered by the VMRTs \mathcal{V}_x for $x \in X$ general. Since \mathcal{K}_x has dimension $n - 2$, the total VMRT \mathcal{V} is a divisor in $\mathbb{P}(\Omega_X)$.

For a general $[l] \in \mathcal{K}$, one has

$$(3.3.1) \quad T_X|_l \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus n-2} \oplus \mathcal{O}_{\mathbb{P}^1}$$

[Deb01, Exercise 4.8.3]. We call a minimal rational curve $[l] \in \mathcal{K}$ *standard* if l is smooth and the bundle $T_X|_l$ has the same splitting type as in (3.3.1).

3.4. Smoothing pairs of minimal curves. For a general point $x_1 \in X$, by the bend-and-break lemma [Deb01, Prop.3.2] the curves parametrised by \mathcal{K}_{x_1} cover a divisor $D_{x_1} \subset X$. This divisor is ample because $\rho(X) = 1$, so for $x_2 \in X$ and $[l_2] \in \mathcal{K}_{x_2}$ the curve l_2 intersects D_{x_1} . Thus for a pair of general points $x_1, x_2 \in X$ we can find a chain of two standard curves $l_1 \cup l_2$ connecting the points x_1 and x_2 .

Let $\text{loc}_{x_1}^1$ be the locus covered by *all* the minimal rational curves of X passing through x_1 . It is itself a divisor, but may be bigger than D_{x_1} since in general there are finitely many families of minimal curves. We choose a pair of general points $x_1, x_2 \in X$ such that $x_2 \notin \text{loc}_{x_1}^1$ (which implies $x_1 \notin \text{loc}_{x_2}^1$), and consider a chain of two standard curves $l_1 \cup l_2$ connecting the points x_1 and x_2 as above.

By [Kol96, II, Ex.7.6.4.1] the union $l_1 \cup l_2$ is dominated by a transverse union $\mathbb{P}^1 \cup \mathbb{P}^1$. Since both rational curves are free we can smooth the tree $\mathbb{P}^1 \cup \mathbb{P}^1$ keeping the point x_1 fixed [Kol96, II, Thm.7.6.1]. Since x_1 is general in X this defines a family of rational curves dominating X , and we denote by \mathcal{W} the normalisation of the irreducible component of $\text{Chow}(X)$ containing these rational curves.

3.5. Since a general member $[C]$ of the family \mathcal{W} is free and $-K_X \cdot C = 2n$, we have $\dim \mathcal{W} = 3n - 3$. We pick an arbitrary irreducible component of the subset of \mathcal{W} parametrising cycles containing x_1 , and let \mathcal{W}_{x_1} be its normalisation; then we have $\dim \mathcal{W}_{x_1} = 2n - 2$. Let \mathcal{U}_{x_1} be the normalisation of the universal family of cycles over \mathcal{W}_{x_1} . The evaluation map $\text{ev}_{x_1} : \mathcal{U}_{x_1} \rightarrow X$ is surjective: its image is irreducible, and it contains both the divisor D_{x_1} (because it is contained in the image of the restriction of ev_{x_1} to those members of \mathcal{W}_{x_1} that contain a minimal curve through x_1) and the point $x_2 \notin D_{x_1}$.

Next, we choose an arbitrary irreducible component of the subset of \mathcal{W} parametrising cycles passing through x_1 and x_2 , and let \mathcal{W}_{x_1, x_2} be its normalisation, \mathcal{U}_{x_1, x_2} the normalisation of the universal family over \mathcal{W}_{x_1, x_2} . We denote by

$$q : \mathcal{U}_{x_1, x_2} \rightarrow \mathcal{W}_{x_1, x_2}, \quad \text{ev} : \mathcal{U}_{x_1, x_2} \rightarrow X$$

the natural maps. It follows from the considerations above that \mathcal{W}_{x_1, x_2} is non-empty of dimension $n - 1$.

By construction, a general curve $[C] \in \mathcal{W}_{x_1, x_2}$ is smooth at x_i , $i \in \{1, 2\}$, so the preimage $\text{ev}^{-1}(x_i)$ contains a unique divisor σ_i that surjects onto \mathcal{W}_{x_1, x_2} . Since ev is finite on the q -fibres and \mathcal{W}_{x_1, x_2} is normal, we obtain that the degree one map $\sigma_i \rightarrow \mathcal{W}_{x_1, x_2}$ is an isomorphism. We call the divisors σ_i the distinguished sections of q .

Let $\Delta \subset \mathcal{W}_{x_1, x_2}$ be the locus parametrising non-integral cycles.

3.6. Lemma. *In the situation of Proposition E and using the notation introduced above, let*

$$C = \sum_8 a_i l_i$$

be a non-integral cycle corresponding to a point $[C] \in \Delta$. Then $C = l_1 + l_2$, with the l_i minimal rational curves such that $x_i \in l_j$ if and only if $i = j$.

Remark. Note that we do not claim that the curves l_i belong to the family \mathcal{K} . However by construction of the family \mathcal{W} as smoothings of pairs $l_1 \cup l_2$ in \mathcal{K} there exists an irreducible component $\Delta_{\mathcal{K}} \subset \Delta$ such that $l_i \in \mathcal{K}$.

Proof. By [Kol96, II, Prop.2.2] all the irreducible components l_i are rational curves. We can suppose that up to renumbering one has $x_1 \in l_1$. If $a_1 \geq 2$, then $-K_X \cdot C = 2n$ and $-K_X \cdot l_1 \geq n$ implies that $C = 2l_1$ and l_1 is a minimal rational curve. Yet this contradicts the assumption $x_2 \notin \text{loc}_{x_1}^1$. Thus we have $a_1 = 1$ and since C is not integral there exists a second irreducible component l_2 . Again $-K_X \cdot C = 2n$ and $-K_X \cdot l_i \geq n$ implies $C = l_1 + l_2$ and the l_i are minimal rational curves by Lemma 3.2. The last property now follows by observing that $x_2 \notin \text{loc}_{x_1}^1$ implies that $x_1 \notin \text{loc}_{x_2}^1$. \square

By [Kol96, II, Thm.2.8], the fibration $q : \mathcal{U}_{x_1, x_2} \rightarrow \mathcal{W}_{x_1, x_2}$ is a \mathbb{P}^1 -bundle over the open set $\mathcal{W}_{x_1, x_2} \setminus \Delta$. Although Lemma 3.6 essentially says that the singular fibres are reducible conics, it is a priori not clear that q is a conic bundle (cf. Definition 2.1). This becomes true after we make a base change to a smooth curve.

3.7. Lemma. *In the situation of Proposition E and using the notation introduced above, let $Z \subset \mathcal{W}_{x_1, x_2}$ be a curve such that a general point of Z parametrises an irreducible curve. Then there exists a finite morphism $T \rightarrow Z$ such that the normalisation S of the fibre product $\mathcal{U}_{x_1, x_2} \times_{\mathcal{W}_{x_1, x_2}} T$ has a conic bundle structure $\varphi : S \rightarrow T$ that satisfies the conditions of Lemma 2.4.*

Proof. Let $\nu : \tilde{Z} \rightarrow Z$ be the normalisation, and let N be the normalisation of $\mathcal{U}_{x_1, x_2} \times_{\mathcal{W}_{x_1, x_2}} \tilde{Z}$, $f_N : N \rightarrow X$ the morphism induced by $\text{ev} : \mathcal{U}_{x_1, x_2} \rightarrow X$. Since all the curves pass through x_1 and x_2 there exists a curve $Z_1 \subset N$ (resp. $Z_2 \subset N$) that is contracted by f_N onto the point x_1 (resp. x_2). Since ev is finite on the q -fibres, the curves Z_1 and Z_2 are multisections of $N \rightarrow \tilde{Z}$. If \tilde{Z}_i is the normalisation of Z_i , then the fibration $(N \times_{\tilde{Z}} \tilde{Z}_i) \rightarrow \tilde{Z}_i$ has a section given by $c \mapsto (c, c)$. Thus there exists a finite base change $T \rightarrow \tilde{Z}$ such that the normalisation $\varphi : S \rightarrow T$ of the fibre product $(\mathcal{U}_{x_1, x_2} \times_{\mathcal{W}_{x_1, x_2}} T) \rightarrow T$ has a natural morphism $f : S \rightarrow X$ induced by $\text{ev} : \mathcal{U}_{x_1, x_2} \rightarrow X$ and contracts two φ -sections σ_1 and σ_2 on x_1 and x_2 respectively.

Since $Z \not\subset \Delta$, the general φ -fibre is \mathbb{P}^1 . Moreover by Lemma 3.6 all the φ -fibres are reduced and have at most two irreducible components. By Lemma 2.3 this implies that φ is a conic bundle and if $s \in S_{\text{sing}}$, then $F_{\varphi(s)}$ is a reducible conic and the two irreducible components meet in s . Thus we have $\sigma_i \subset S_{\text{non-s}}$, since otherwise both irreducible components would pass through x_i , thereby contradicting the property that $x_2 \notin \text{loc}_{x_1}^1$. For the same reason we can decompose any reducible φ -fibre F_t by defining $F_{t,i}$ as the unique component meeting the section σ_i . Since $\sigma_i \cdot F = 1$ for a general φ -fibre we see that (C1) holds. Condition (C2) holds with H the pull-back of an ample divisor on X . \square

From this one deduces with Lemma 2.4 the following statement, in the spirit of the bend-and-break lemma [Deb01, Prop.3.2].

3.8. Lemma. *The restriction of the evaluation map $\text{ev} : \mathcal{U}_{x_1, x_2} \rightarrow X$ to the complement of $\sigma_1 \cup \sigma_2$ is quasi-finite. In particular ev is generically finite onto its image.*

Proof. We argue by contradiction. Since ev is finite on the q -fibres there exists a curve $Z \subset \mathcal{W}_{x_1, x_2}$ such that the natural map from the surface $q^{-1}(Z)$ onto $\text{ev}(q^{-1}(Z))$ contracts three disjoint curves σ_1, σ_2 and σ onto the points x_1, x_2 and x .

If $Z \not\subset \Delta$, then by Lemma 3.7 we can suppose, possibly up to a finite base change, that $q^{-1}(Z) \rightarrow Z$ satisfies the conditions (C1) of Lemma 2.4. After a further base change we can assume that σ is a section. Since σ is contracted by ev we have $\sigma^2 < 0$. By Lemma 2.4,a), this implies $\sigma = \sigma_1$ or $\sigma = \sigma_2$, a contradiction.

If $Z \subset \Delta$, then all the fibres over Z are unions of two minimal rational curves. Thus the normalisation of $q^{-1}(Z)$ is a union of two \mathbb{P}^1 -bundles mapping onto Z and by construction they contain three curves which are mapped onto points. However a ruled surface contains at most one contractible curve, a contradiction. \square

3.9. Since $\dim \mathcal{U}_{x_1, x_2} = \dim X$, one deduces from Lemma 3.8 above that the cycles $[C] \in \mathcal{W}$ passing through x_1, x_2 cover the manifold X . By [Deb01, 4.10] this implies that a general member $[C] \in \mathcal{W}_{x_1, x_2}$ is a 2-free rational curve [Deb01, Defn.4.5]. Since $-K_X \cdot C = 2n$, this forces

$$(3.9.1) \quad f^*T_X \simeq \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus n},$$

where $f : \mathbb{P}^1 \rightarrow C \subset X$ is the normalisation of C . As a consequence, one sees from [Kol96, II, Thm.3.14.3] that a general member $[C] \in \mathcal{W}$ is a *smooth* rational curve in X .

Let $\text{Hom}_{\mathcal{W}}^\circ \subset \text{Hom}(\mathbb{P}^1, X)$ be the irreducible open set parametrising morphisms $f : \mathbb{P}^1 \rightarrow X$ such that the image $C := f(\mathbb{P}^1)$ is smooth, the associated cycle $[C] \in \text{Chow}(X)$ is a point in \mathcal{W} , and f^*T_X has the splitting type (3.9.1). By what precedes, the image of $\text{Hom}_{\mathcal{W}}^\circ$ in \mathcal{W} under the natural map $\text{Hom}(\mathbb{P}^1, X) \rightarrow \text{Chow}(X)$ is a dense open set $\mathcal{W}^\circ \subset \mathcal{W}$.

3.10. Denote by $\pi : \mathbb{P}(\Omega_X) \rightarrow X$ the projection map. We define an injective map

$$i : \text{Hom}_{\mathcal{W}}^\circ \hookrightarrow \text{Hom}(\mathbb{P}^1, \mathbb{P}(\Omega_X))$$

by mapping $f : \mathbb{P}^1 \rightarrow X$ to the morphism $\tilde{f} : \mathbb{P}^1 \rightarrow \mathbb{P}(\Omega_X)$ corresponding to the invertible quotient $f^*\Omega_X \rightarrow \Omega_{\mathbb{P}^1}$. Correspondingly, for $[C] \in \mathcal{W}^\circ$ with normalisation f , we call $[\tilde{C}]$ the member of $\text{Chow}(\mathbb{P}(\Omega_X))$ corresponding to the lifting \tilde{f} .

We let $\text{Hom}_{\mathcal{W}}^\sim$ be the image of i . Note that it parametrises a family of rational curves that dominates $\mathbb{P}(\Omega_X)$, but it is not an irreducible component of $\text{Hom}(\mathbb{P}^1, \mathbb{P}(\Omega_X))$. Indeed, $\text{Hom}_{\mathcal{W}}^\sim$ is contained in a (much bigger) irreducible component defined by morphisms corresponding to arbitrary quotients $f^*\Omega_X \rightarrow \mathcal{O}_{\mathbb{P}^1}(-2)$.

The following property is well-known to experts. Since $\text{Hom}_{\mathcal{W}}^\sim$ is not an open set of the Hom-space $\text{Hom}(\mathbb{P}^1, \mathbb{P}(\Omega_X))$, we have to adapt the proof of [Kol96, II, Prop.3.7].

3.11. Lemma. *In the situation of Proposition E, let $\mathcal{V}_0 \subset \mathcal{V}$ be a dense, Zariski open set in the total VMRT \mathcal{V} , and let $\tilde{C} := \tilde{f}(\mathbb{P}^1)$ be a rational curve parametrised*

by a general point of $\mathrm{Hom}_{\tilde{\mathcal{W}}}$. Then one has

$$(\mathcal{V} \cap \tilde{C}) \subset (\mathcal{V}_0 \cap \tilde{C}).$$

Proof. Set $Z := \mathcal{V} \setminus \mathcal{V}_0$. A point $z \in \mathbb{P}(\Omega_X)$ is $z = (v_z^\perp, x)$, where $\mathbb{C}v_z \subset T_{X,x}$ is a tangent direction in X at $x = \pi(z)$. So for all $p \in \mathbb{P}^1$, $z = (v_z^\perp, x) \in \mathbb{P}(\Omega_X)$, the morphisms $[\tilde{f}] \in \mathrm{Hom}_{\tilde{\mathcal{W}}}$ mapping p to z correspond to morphisms $f : \mathbb{P}^1 \rightarrow X$ in $\mathrm{Hom}_{\mathcal{W}}^\circ$ mapping p to x with tangent direction $\mathbb{C}v_z$. Since f has the splitting type (3.9.1), the set of these morphisms has dimension exactly n . It follows that

$$\mathrm{Hom}_{\tilde{\mathcal{W}},Z} := \{[\tilde{f}] \in \mathrm{Hom}_{\tilde{\mathcal{W}}} \mid \tilde{f}(\mathbb{P}^1) \cap Z \neq \emptyset\} = \bigcup_{z \in Z} \bigcup_{p \in \mathbb{P}^1} \{[\tilde{f}] \in \mathrm{Hom}_{\tilde{\mathcal{W}}} \mid \tilde{f}(p) = z\}$$

has dimension at most $\dim Z + 1 + n$.

Now $\mathcal{V} \subset \mathbb{P}(\Omega_X)$ is a divisor, and Z has codimension at least one in \mathcal{V} , so Z has dimension at most $2n - 3$, and the set $\mathrm{Hom}_{\tilde{\mathcal{W}},Z}$ above has dimension at most $3n - 2$. Since $\mathrm{Hom}_{\mathcal{W}}^\circ$ has dimension $3n$ and $\mathrm{Hom}_{\mathcal{W}}^\circ \rightarrow \mathrm{Hom}_{\tilde{\mathcal{W}}}$ is injective, a general point $[\tilde{f}] \in \mathrm{Hom}_{\tilde{\mathcal{W}}}$ is not in $\mathrm{Hom}_{\tilde{\mathcal{W}},Z}$. \square

We need one more technical statement:

3.12. Lemma. *In the situation of Proposition E and using the notation introduced above, let $[f] \in \mathrm{Hom}_{\mathcal{W}}^\circ$ be a general point. Then for every $x \in f(\mathbb{P}^1)$ we have $f(\mathbb{P}^1) \not\subset \mathrm{loc}_x^1$.*

Proof. Fix two general points $x_1, x_2 \in X$. A general morphism $[f] \in \mathrm{Hom}_{\mathcal{W}}^\circ$ passing through x_1 and x_2 is 2-free and up to reparametrisation we have $f(0) = x_1, f(\infty) = x_2$. Set $g := f|_{\{0,\infty\}}$, then f is free over g [Kol96, II, Defn.3.1]. Suppose now that such a curve has the property $f(\mathbb{P}^1) \subset \mathrm{loc}_{x_0}^1$ for some $x_0 \in f(\mathbb{P}^1)$. Thus $x_1, x_2 \in \mathrm{loc}_{x_0}^1$, hence by symmetry $x_0 \in (\mathrm{loc}_{x_1}^1 \cap \mathrm{loc}_{x_2}^1)$. Yet the intersection

$$\mathrm{loc}_{x_1}^1 \cap \mathrm{loc}_{x_2}^1$$

has codimension two in X . By [Kol96, II, Prop.3.7] a general deformation of f over g is disjoint from this set. \square

3.13. Proof of Proposition E. Arguing by contradiction, we suppose that $\mathcal{V} \cdot \tilde{C} > 0$ (\tilde{C} is not contained in \mathcal{V} for the general $[C] \in \mathcal{W}^\circ$). Applying Lemma 3.11 with

$$\mathcal{V}_0 := \{v^\perp \in \mathcal{V} \mid \mathbb{C}v = T_{l,\pi(v)} \text{ where } [l] \in \mathcal{K} \text{ is standard}\},$$

we see that for a general point $[C] \in \mathcal{W}$ there exists a point $x_1 \in C$ and a standard curve $[l] \in \mathcal{K}_{x_1}$ such that

$$(3.13.1) \quad T_{C,x_1} = T_{l,x_1}.$$

We shall now reformulate the property (3.13.1) in terms of the universal family \mathcal{U}_{x_1,x_2} , with x_2 a point chosen in $C \setminus \mathrm{loc}_{x_1}^1$ thanks to Lemma 3.12. Consider the blow-up $\varepsilon : \tilde{X} \rightarrow X$ at the point x_1 , with exceptional divisor E_1 . Since the general member of \mathcal{W}_{x_1,x_2} is smooth at x_1 , there is a rational map $\mathrm{ev} : \mathcal{U}_{x_1,x_2} \dashrightarrow \tilde{X}$ such that $\varepsilon \circ \mathrm{ev} = \mathrm{ev}$ (on the locus where ev is defined). It restricts to a well-defined rational map $\sigma_1 \dashrightarrow E_1$ which is dominant, and therefore generically finite, because the general member of \mathcal{W}_{x_1,x_2} is 2-free. In particular we may assume it is finite in a neighbourhood of the point $C \cap \sigma_1$.

We then consider the proper transform \tilde{l} of l under ε , and let Γ be an irreducible component of $\tilde{\text{ev}}^{-1}(\tilde{l})$ passing through $C \cap \sigma_1$. It is a curve that is mapped to a curve in \mathcal{W}_{x_1, x_2} by q . Also, applying the same construction to the divisor $D_{x_1} \subset X$, one gets a prime divisor $G \subset \mathcal{U}_{x_1, x_2}$ mapping surjectively onto D_{x_1} and \mathcal{W}_{x_1, x_2} respectively.

Since both maps $q|_G$ and $\text{ev}|_G$ are étale at the general point of G , for the general $l' \in \mathcal{K}_{x_1}$ there exists a curve $\Gamma' \subset G$ such that $q(\Gamma')$ is a curve, $\text{ev}(\Gamma') = l'$, both maps $q|_G$ and $\text{ev}|_G$ are étale at the general point $x \in l'$, and there is $[C'] \in \mathcal{W}^\circ$ such that the point $C' \cap \sigma_1$ lies on Γ' . Yet this is a contradiction to Proposition 3.14 below. \square

3.14. Proposition [Miy04, Lemma 3.9]. *In the situation of Proposition E, let $x_1, x_2 \in X$ be general points, and $[l]$ a general member of \mathcal{K}_{x_1} . Consider an irreducible curve $\Gamma \subset \mathcal{U}_{x_1, x_2}$ such that $\text{ev}(\Gamma) = l$ and $q(\Gamma)$ is a curve, and assume there exists a prime divisor $G \subset \mathcal{U}_{x_1, x_2}$ mapped onto D_{x_1} by ev and containing Γ , such that both maps $q|_G$ and $\text{ev}|_G$ are étale at a general point of Γ . Then $\Gamma \cap \sigma_1$ does not contain any point $C \cap \sigma_1$ with $[C] \in \mathcal{W}^\circ$.*

We give the proof for the sake of completeness.

Proof. Since $[l]$ is general in \mathcal{K}_{x_1} , we have

$$T_X|_l \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{n-2} \oplus \mathcal{O}_{\mathbb{P}^1},$$

and \mathcal{K}_{x_1} is smooth with tangent space $H^0(l, N_{l/X}^+ \otimes \mathcal{O}_l(-x_1))$ at $[l]$, where \mathcal{E}^+ denotes the ample part of a vector bundle $\mathcal{E} \rightarrow \mathbb{P}^1$, i.e. its ample subbundle of maximal rank.

Let $x \in \Gamma$ be a general point, and set $y = \text{ev}(x) \in l$. For some analytic neighbourhood $V \subset \mathcal{K}_{x_1}$ of $[l]$, we have an evaluation map

$$\mathbb{P}^1 \times V \longrightarrow D_{x_1}$$

which is étale at $(y, [l])$, and the tangent space to D_{x_1} at y is thus

$$T_{D_{x_1}, y} = T_{l, y} \oplus (N_{l/X}^+ \otimes \mathcal{O}_l(-x_1))_y = T_X|_{l, y}^+.$$

Since $q|_G$ and $\text{ev}|_G$ are étale in x we obtain that

$$(3.14.1) \quad (q^*T_{\mathcal{W}_{x_1, x_2}})_x = \text{ev}^*T_X|_{l, \text{ev}(x)}^+$$

as subspaces of $T_{\mathcal{U}_{x_1, x_2}, x}$.

We argue by contradiction and suppose that there exists $[C] \in \mathcal{W}^\circ$ such that $(C \cap \sigma_1) \in (\Gamma \cap \sigma_1)$. Since Γ maps onto l it is not contained in the divisor σ_1 . Since the smooth rational curve C is 2-free, there exists by semicontinuity a neighbourhood U of $[C] \in \mathcal{W}_{x_1, x_2}$ parametrising 2-free smooth rational curves. For a 2-free rational curve, the evaluation morphism ev is smooth in the complement of the distinguished divisors σ_i [Kol96, II, Prop.3.5.1]. Thus if we denote by $R \subset \mathcal{U}_{x_1, x_2}$ the ramification divisor of ev , σ_1 is the unique irreducible component of R containing the point $C \cap \sigma_1$. Thus Γ is not contained in the ramification divisor of ev .

Since $q(\Gamma)$ is a curve, there exists by Lemma 3.7 a finite base change $T \rightarrow q(\Gamma)$ with T a smooth curve, such that the normalisation S of the fibre product $T \times_{\mathcal{W}_{x_1, x_2}} \mathcal{U}_{x_1, x_2}$ is a surface with a conic bundle structure $\varphi : S \rightarrow T$ satisfying the conditions of Lemma 2.4. After a further base change we may suppose that

there exists a φ -section Γ_1 that maps onto Γ . Note that since we obtained S by a base change from \mathcal{U}_{x_1, x_2} , the ramification divisor of the map $\mu : S \rightarrow \mathcal{U}_{x_1, x_2}$ is contained in the φ -fibres, i.e. its image by φ has dimension 0. In particular Γ_1 is not contained in this ramification locus.

Since the rational curve C is smooth and 2-free, the universal family \mathcal{U}_{x_1, x_2} is smooth in a neighbourhood of $C \cap \sigma_1$. Thus σ_1 is a Cartier divisor in a neighbourhood of $C \cap \sigma_1$, and we can use the projection formula to see that

$$\Gamma_1 \cdot \mu^* \sigma_1 = \mu_*(\Gamma_1) \cdot \sigma_1 > 0.$$

In particular Γ_1 is not disjoint from the distinguished sections in the conic bundle $S \rightarrow T$. Let now $\varepsilon : \hat{S} \rightarrow S$ be the minimal resolution of singularities, and $\hat{\Gamma}_1$ the proper transform of Γ_1 . Since the distinguished sections are in the smooth locus of S , the section $\hat{\Gamma}_1$ is not disjoint from the distinguished sections of $\hat{S} \rightarrow T$. We shall now show that

$$(\hat{\Gamma}_1)^2 \leq 0,$$

which is a contradiction to Lemma 2.4.

Denote by $f : \hat{\Gamma}_1 \rightarrow l$ the restriction of $\text{ev} \circ \mu \circ \varepsilon : \hat{S} \rightarrow X$. Since $\hat{\Gamma}_1$ is not in the ramification locus of $\mu \circ \varepsilon$ and Γ is not in the ramification divisor of ev , the tangent map

$$T_{\hat{S}}|_{\hat{\Gamma}_1} \rightarrow f^* T_X|_l$$

is generically injective. Since $\hat{\Gamma}_1$ is a $\varphi \circ \varepsilon$ -section, we have an isomorphism

$$(3.14.2) \quad T_{\hat{S}/T}|_{\hat{\Gamma}_1} \simeq N_{\hat{\Gamma}_1/\hat{S}}.$$

Since l has the standard splitting type (3.3.1) we have a (unique) trivial quotient $f^* T_X|_l \twoheadrightarrow \mathcal{O}_{\hat{\Gamma}_1}$, and thanks to (3.14.2) we are done if we prove that the natural map

$$T_{\hat{S}/T}|_{\hat{\Gamma}_1} \hookrightarrow T_{\hat{S}}|_{\hat{\Gamma}_1} \rightarrow f^* T_X|_l \twoheadrightarrow \mathcal{O}_{\hat{\Gamma}_1}$$

is not zero. It is sufficient to check this property for a general point in $\hat{\Gamma}_1$, and since $\hat{\Gamma}_1 \rightarrow \Gamma$ is generically étale, it is sufficient to check that for a general $x \in \Gamma$, the natural map

$$T_{\mathcal{U}_{x_1, x_2}/\mathcal{W}_{x_1, x_2}, x} \rightarrow \text{ev}^*(T_{X, \text{ev}(x)})$$

does not have its image into the ample part $\text{ev}^*(T_X|_{l, \text{ev}(x)}^+)$. Yet by (3.14.1) we know that $(q^* T_{\mathcal{W}_{x_1, x_2}})_x$ maps into the ample part. Thus if $T_{\mathcal{U}_{x_1, x_2}/\mathcal{W}_{x_1, x_2}, x}$ also maps into the ample part, then the tangent map

$$T_{\mathcal{U}_{x_1, x_2}, x} \rightarrow \text{ev}^*(T_{X, \text{ev}(x)})$$

cannot be surjective. Since Γ is not contained in the ramification locus of ev this is a contradiction. \square

4. PROOF OF THE MAIN THEOREM

4.1. Proof of Theorem C. If $X \simeq \mathbb{P}^n$ we are done, so suppose that this is not the case. Then consider the family of minimal rational curves \mathcal{K} constructed in Section 3 and the associated total VMRT \mathcal{V} . Denote by $d \in \mathbb{N}$ the degree of a general VMRT $\mathcal{V}_x \subset \mathbb{P}(\Omega_{X, x})$.

Step 1. Using the family \mathcal{W}° . In this step we prove that

$$(4.1.1) \quad \mathcal{V} \sim_{\mathbb{Q}} d(\zeta - \frac{1}{n}\pi^*K_X),$$

where ζ is the tautological divisor class on $\mathbb{P}(\Omega_X)$. Note that $\mathbb{P}(\Omega_X)$ has Picard number two, so we can always write

$$\mathcal{V} \sim_{\mathbb{Q}} a\zeta + b\frac{-1}{n}\pi^*K_X$$

with $a, b \in \mathbb{Q}$. Let now \mathcal{W}° be the family of rational curves constructed in Section 3, and let \tilde{C} be the lifting of a curve $C \in \mathcal{W}^\circ$. By Proposition E we have $\mathcal{V} \cdot \tilde{C} = 0$. Since by the definition of \tilde{C} one has $\zeta \cdot \tilde{C} = -2$ and $-\frac{1}{n}\pi^*K_X \cdot \tilde{C} = 2$, it follows that $a = b$. Since $\mathcal{V}_x = \mathcal{V}|_{\mathbb{P}(\Omega_{X,x})} \sim_{\mathbb{Q}} d\zeta|_{\mathbb{P}(\Omega_{X,x})}$, we have $a = b = d$. This proves (4.1.1).

Step 2. Bounding the degree d . Denote by $\mathcal{K}^\circ \subset \mathcal{K}$ the open set parametrising smooth standard rational curves in \mathcal{K} . We define an injective map

$$j : \mathcal{K}^\circ \hookrightarrow \text{RatCurves}^n(\mathbb{P}(\Omega_X))$$

by mapping a curve l to the image \tilde{l} of the morphism $s : l \rightarrow \mathbb{P}(\Omega_X)$ defined by the invertible quotient $\Omega_X|_l \rightarrow \Omega_l$. We denote by $\tilde{\mathcal{K}}^\circ$ the image of j . Let us start by showing that $\tilde{\mathcal{K}}^\circ$ is dense in an irreducible component of $\text{RatCurves}^n(\mathbb{P}(\Omega_X))$. Arguing by contradiction we suppose that $\tilde{\mathcal{K}}^\circ$ is contained in an irreducible component \mathcal{R} of dimension strictly larger than $2n - 3$. The projection π defines a map π_* between the spaces of rational curves, and by construction $\pi_*(\mathcal{R})$ contains \mathcal{K}° . Since \mathcal{K}° is dense in an irreducible component of $\text{RatCurves}^n(X)$ we obtain that (up to replacing \mathcal{R} by a Zariski open set) we have a map $\pi_* : \mathcal{R} \rightarrow \mathcal{K}^\circ$. Since $\dim \mathcal{R} > 2n - 3$ this map has positive-dimensional fibres, so a general curve \tilde{l} deforms in $\mathbb{P}(\Omega_X|_l)$. Yet this is impossible since

$$N_{\tilde{l}/\mathbb{P}(\Omega_X|_l)} \simeq \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus n-2}.$$

By construction the lifted curves \tilde{l} are contained in \mathcal{V} . Thus the open set $\tilde{\mathcal{K}}_0 \subset \text{RatCurves}^n(\mathbb{P}(\Omega_X))$ is actually an open set in $\text{RatCurves}^n(\mathcal{V})$. Since $\mathcal{V} \subset \mathbb{P}(\Omega_X)$ is a hypersurface, the algebraic set \mathcal{V} has lci singularities. Thus we can apply [Kol96, II, Thm.1.3, Thm.2.15] and obtain

$$2n - 3 = \dim \tilde{\mathcal{K}}_0 \geq -K_{\mathcal{V}} \cdot \tilde{l} + (2n - 2) - 3.$$

We thus have $-K_{\mathcal{V}} \cdot \tilde{l} \leq 2$.

Now by construction we have $-\frac{1}{n}\pi^*K_X \cdot \tilde{l} = 1$ and $\zeta \cdot \tilde{l} = -2$. Since $K_{\mathbb{P}(\Omega_X)} = 2\pi^*K_X - n\zeta$, the adjunction formula and (4.1.1) yield

$$2 \geq -K_{\mathcal{V}} \cdot \tilde{l} = -(K_{\mathbb{P}(\Omega_X)} + \mathcal{V}) \cdot \tilde{l} = d.$$

Step 3. Conclusion. If $d = 1$ or $d = 2$ but \mathcal{V}_x is reducible, we obtain a contradiction to [Hwa07, Thm.1.5] (cf. also [Ara06, Thm.3.1]). If $d = 2$ and \mathcal{V}_x is irreducible, \mathcal{V}_x is normal [Har77, II, Ex.6.5(a)], and therefore isomorphic to its normalisation \mathcal{K}_x which is smooth (see §3.3). It is thus a smooth quadric and we conclude by [Mok08, Main Thm.]. \square

4.2. Remark. Let us explain the difference of our proof with Miyaoka's approach: in the notation of Section 3, he considers the family \mathcal{W}_{x_1, x_2} . As we have seen above the evaluation map $\text{ev} : \mathcal{U}_{x_1, x_2} \rightarrow X$ is generically finite and his goal is to

prove that ev is birational. He therefore analyses the preimage $\text{ev}^{-1}(l_1 \cup l_2)$, where the $l_i \subset X$ are general minimal curves passing through x_i respectively such that $[l_1 \cup l_2] \in \mathcal{W}_{x_1, x_2}$. If $\Gamma \subset \text{ev}^{-1}(l_1 \cup l_2)$ is an irreducible curve mapping onto l_1 one can make a case distinction: if $q(\Gamma)$ is a curve that is not contained in the discriminant locus $\Delta \subset \mathcal{W}_{x_1, x_2}$ (Case **C** in [Miy04, p.227]) Miyaoka makes a very interesting observation which we stated as Proposition 3.14. However the analysis of the ‘trivial’ case (Case **A** in [Miy04, p.227]) where $q(\Gamma)$ is a point is not correct: it is not clear that $q(\Gamma) = [l_1 \cup l_2]$, because there might be another curve in \mathcal{W}_{x_1, x_2} which is of the form $l_1 \cup l'_2$ with $l_2 \neq l'_2$. This possibility is an obvious obstruction to the birationality of ev and invalidates [Miy04, Cor.3.11(2), Cor.3.13(1)]. The following example shows that this possibility does indeed occur in certain cases.

4.3. Example. Let $H \subset \mathbb{P}^n$ be a hyperplane and $A \subset H \subset \mathbb{P}^n$ a projective manifold A of dimension $n - 2$ and degree $3 \leq a \leq n$. Let $\mu : X \rightarrow \mathbb{P}^n$ be the blow-up of \mathbb{P}^n along A . Then X is a Fano manifold [Miy04, Rem.4.2] and $-K_X \cdot C \geq n$ for every rational curve $C \subset X$ passing through a *general* point (the μ -fibres are however rational curves with $-K_X \cdot C = 1$). The general member of a family of minimal rational curves \mathcal{K} is the proper transform of a line that intersects A . Consider the family \mathcal{W} whose general member is the strict transform of a reduced, connected degree two curve C such that $A \cap C$ is a finite scheme of length two. For general points $x_1, x_2 \in X$ the (normalised) universal family $\mathcal{U}_{x_1, x_2} \rightarrow \mathcal{W}_{x_1, x_2}$ is a conic bundle and the evaluation map $\text{ev} : \mathcal{U}_{x_1, x_2} \rightarrow X$ is generically finite. We claim that ev is not birational.

Proof of the claim. For simplicity of notation we denote by x_1, x_2 also the corresponding points in \mathbb{P}^n . Let $l_1 \subset \mathbb{P}^n$ be a general line through x_1 that intersects A . Since $x_2 \in \mathbb{P}^n$ is general there exists a unique plane Π containing l_1 and x_2 . Moreover the intersection $\Pi \cap A$ consists of exactly a points, one of them the point $A \cap l_1$. For every point $x \in \Pi \cap A$ other than $A \cap l_1$, there exists a unique line $l_{2,x}$ through x and x_2 . By Bezout’s theorem $l_1 \cup l_2$ is connected, so its proper transform belongs to \mathcal{W}_{x_1, x_2} . Yet this shows that $\text{ev}^{-1}(l_1)$ contains $a - 1 > 1$ copies of l_1 , one for each point $x \in \Pi \cap A \setminus l_1 \cap A$. This proves the claim. \square

Let us conclude this example by mentioning that the conic bundle $\mathcal{U}_{x_1, x_2} \rightarrow \mathcal{W}_{x_1, x_2}$ does not satisfy the symmetry conditions of Lemma 2.4.

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